REALCOMPACTNESS AND MEASURE-COMPACTNESS OF THE UNIT BALL IN A BANACH SPACE*

G.A. Edgar

Department of Mathematics

The Ohio State University

Columbus, Ohio 43210 U.S.A.

Abstract. It is proved that the unit ball (with its weak topology) is not realcompact in the Banach spaces ℓ_∞/c_0 and $J(\omega_1)$. It is stated, but not proved, that the unit ball is not measure-compact in the Banach space ℓ_∞ .

1. Let X be a Banach space. Topological properties of the weak topology $\sigma(X, X^*)$ have been of interest recently (for example [4][9]). The unit ball $B_X = \{x \in X : ||x|| \leq 1\}$ in the relative weak topology can also be considered. Since $(B_X$, weak) is a closed subset of $(X$, weak), we see that if $(X$, weak) is realcompact (measure-compact), so is $(B_X$, weak) . The question I will be concerned with in this paper is whether the converse is true.

I do not have an answer to the question in general. In this paper, some concrete Banach spaces X are considered that are known not to be realcompact (or measure-compact), and it is proved that B_X is also not realcompact (or measure-compact). In some cases this is more difficult for B_X than for X . Reasons for the extra difficulty are hard to pin down. Corson's criterion for realcompactness in X [1, p. 10] is false when applied to B_X (see Theorem 5.3, below). The σ-algebra of Baire sets for X is generated by X^* [4, Theorem 2.3] but this is not necessarily true for B_X (see Section 3).

Topological words and phrases will always refer to the weak topology $\sigma(X, X^*)$ unless the contrary is specified. If T is a topological space, we write $C(T)$ for the set of all continuous, real-valued functions on T .

General background on realcompactness can be found in [8]; on measure-compactness can be found in [9].

2. In this preliminary section, we will recast some topological conditions in terms of nets. Doubtless this could be avoided in the sequel, but I find it helpful.

*Supported in part by National Science Foundation grant MCS 8003078.

2.1 Definition. A σ-directed set is a directed set such that every countable subset has an upper bound. A σ-net is a net whose domain is a σ-directed set.

The proofs of the following observations are omitted.

A topological space T is Lindelof if and only if every σ-net in T has a cluster point.

A σ-net that converges in $\mathbb{R}$ is eventually constant. A σ-net in $\mathbb{R}$ that does not converge has at least two finite cluster points.

If a σ-net is in a countable union $\bigcup_{n=1}^{\infty} A_n$, then it is frequently in A_n (for some n).

Let I be a set whose cardinal is not 2-valued measurable [that is, the discrete space I is realcompact]. If (x_ξ) is a σ-net in a union $\bigcup_{i \in I} A_i$ that is not eventually in any A_i, then there exist disjoint $I_1, I_2 \subseteq I$ such that (x_ξ) is frequently in each of the sets $\bigcup_{i \in I_1} A_i$, $\bigcup_{i \in I_2} A_i$.

Let T be a topological space. Then T is realcompact if and only if each σ-net (x_ξ) such that $h(x_\xi)$ converges for all continuous $h : T \to \mathbb{R}$ is convergent. (In general, the limits of such nets are the points of the Hewitt real compactification υX.)

3. I include here an example where Baire $(B_X$, weak$) \neq$ Baire $(X$, weak$) \cap B_X$. Some of the later examples have the same property, but the verification is simpler in this case.

Let $X = \ell_1(\Gamma)$, where card $\Gamma > 2^{\aleph_0}$. Define

$$G = \{f : ||f|| \leq 1 , f(j) > \frac{3}{4} \text{ for some } \gamma \in \Gamma\} .$$

Then (1) G is a cozero set in B_X; and (2) there is no Baire set D in X with $D \cap B_X = G$.

To see that (1) is true, consider the function.

$$f \mapsto \frac{3}{4} \vee \max_{\gamma \in \Gamma} f(\gamma)$$

on $(B_X$, weak$)$. It is continuous since the closure of any set $A_\gamma = \{f : f(\gamma) > \frac{3}{4}\}$ is disjoint from the closure of the union of all the rest.

For (2), suppose D is a Baire set in (X, weak) with $D \cap B_X = G$. Then [4, Theorem 2.3] D is determined by countably many linear functionals $\{g_1, g_2, \dots\} \subseteq \ell_1(\Gamma)^*$. Let e_γ be the canonical unit vectors in $\ell_1(\Gamma)$. Since card $\Gamma > 2^{\aleph_0}$, there is an uncountable $\Gamma_0 \subseteq \Gamma$ with $g_i(e_\gamma) = g_i(e_{\gamma'})$ for all $\gamma, \gamma' \in \Gamma_0$ and all $i = 1, 2, \dots$. Now $e_\gamma \in G \subseteq D$, so $\frac{1}{2}(e_\gamma + e_{\gamma'}) \in D$ when $\gamma, \gamma' \in \Gamma_0$, but not in G. So $D \cap B_X \neq G$.

4. The next example is the space $X = \ell_\infty/c_0$, which Corson showed is not realcompact [1, p. 12]. The proof that B_X is not realcompact is similar to Corson's proof, but greater care must be taken, since Corson's criterion for realcompactness of X may fail for B_X.

We consider $\ell_\infty/c_0 = C(\beta\mathbb{N}\setminus\mathbb{N})$. For countable ordinals α, there exist clopen sets T_α in $\beta\mathbb{N}\setminus\mathbb{N}$ such that if $\alpha < \beta$ then $T_\alpha \subsetneqq T_\beta$ [1, p. 13]. Let $x_\alpha = \chi_{T_\alpha} \in C(\beta\mathbb{N}\setminus\mathbb{N}) = X$, and $F = \chi_{\cup T_\alpha} \in X^{**}$. Corson showed $F \notin X$ but $x_\alpha \to F$ in υX. In fact, $\|x_\alpha\| = 1$, $\|F\| = 1$, so I must show that $h(x_\alpha)$ converges for any $h \in C(B_X)$. Suppose not. Then there exist $a < b$ such that $h(x_\alpha) > b$ frequently and $h(x_\alpha) < a$ frequently.

Note that if $H \subseteq \beta\mathbb{N}\setminus\mathbb{N}$ is the support of a measure, then (by countable additivity) there exists $\beta < \omega_1$ such that $H \cap (\cup_\alpha T_\alpha) = H \cap T_\beta$. So for each α such that $h(x_\alpha) > b$ [respectively, $h(x_\alpha) < a$], choose a basic neighborhood of x_α so that $h(x) > b$ [respectively, $h(x) < a$] on it. By considering finitely many supports of measures, it follows that there exists $\bar{\alpha} < \omega_1$ so that if $x|T_{\bar{\alpha}} = x_\alpha|_{T_{\bar{\alpha}}}$ then $h(x) > b$ [resp., $h(x) < a$]. So, we can choose ordinals $\alpha_1 < \alpha_2 < \dots$ such that $h(x_{\alpha_k}) > b$ for k odd, $h(x_{\alpha_k}) < a$ for k even, $\alpha_{k+1} > \alpha_k$, $\alpha_{k+1} > \bar{\alpha}_k$. Choose $\beta > \sup_k \alpha_k$. Let $y_k = x_{\alpha_k} - x_{\alpha_{k+1}} + x_\beta$. Then $y_k|_{T_{\bar{\alpha}_k}} = x_{\alpha_k}|_{T_{\bar{\alpha}_k}}$, so $h(y_k)$ does not converge. But $\|y_k\| = 1$ so $y_k \in B_X$ and $y_k \to x_\beta$ (pointwise on $\beta\mathbb{N}\setminus\mathbb{N}$ and hence weakly in $C(\beta\mathbb{N}\setminus\mathbb{N})$ by the dominated convergence theorem). So h is not continuous on $C(B_X)$.

5. The next example is the long James space $X = J(\omega_1)$. Notation will be the same as in [6], which I assume is familiar to the reader. Write $B = B_X$.

<u>5.1 THEOREM.</u> If $\mathcal{U}$ is a discrete family of nonempty open sets in $3B$, then

$\{U \in \mathcal{U} : U \cap B \neq \phi\}$ is countable.

<u>Proof.</u> Begin with the following observation: if $\alpha < \omega_1$, and $\mathcal{U}$ is an uncountable family of nonempty open sets in B, then (since $J(\alpha)$ is separable) there exists $f \in B$ such that

$$\{U \in \mathcal{U} : \text{there exists } g \in U,\ g|_{[0,\alpha]\cup\{\omega_1\}} = f|_{[0,\alpha]\cup\{\omega_1\}}\}$$

is uncountable.

Suppose $\mathcal{U}_0 = \{U \in \mathcal{U} : U \cap B \neq \phi\}$ is uncountable. Let $\alpha_0 = 1$. Then there exists $f_1 \in B$ such that

$$\mathcal{U}_1 = \{U \in \mathcal{U}_0 : \text{there exists } g \in U,\ g|_{[0,\alpha_0]\cup\{\omega_1\}} = f_1|_{[0,\alpha_0]\cup\{\omega_1\}}\}$$

is uncountable. Choose $U_1 \in \mathcal{U}_1$. Then choose α_1 so that: $\alpha_1 > \alpha_0$, f_1 is constant on $[\alpha_1, \omega_1]$, and if $f = f_1$ on $[0, \alpha_1]$ then $f \in U_1$. Continue recursively. If α_k, f_k, $\mathcal{U}_k$, U_k have been chosen, there exists $f_{k+1} \in B$ such that $f_k = f_{k+1}$ on $[0, \alpha_{k-1}] \cup \{\omega_1\}$ and

$$\mathcal{U}_{k+1} = \{U \in \mathcal{U}_k : \text{there exists } g \in U,\ g|_{[0,\alpha_k]\cup\{\omega_1\}} = f_{k+1}|_{[0,\alpha_k]\cup\{\omega_1\}}\}$$

is countable. Choose $U_{k+1} \in \mathcal{U}_{k+1}$ different from $U_1, \ldots, U_k$. Then choose α_{k+1} so that: $\alpha_{k+1} > \alpha_k$, f_{k+1} is constant on $[\alpha_{k+1}, \omega_1]$, if $f = f_{k+1}$ on $[0, \alpha_{k+1}]$, then $f \in U_{k+1}$. This completes the recursive construction.

Now let $\beta = \sup \alpha_k$. Define $g : [0, \omega_1] \to \mathbb{R}$ by $g(\alpha) = \lim_k f_k(\alpha)$. So in fact, $g(\alpha) = f_k(\alpha)$ if $\alpha \leq \alpha_{k-1}$, and $g(\alpha) = f_1(\omega_1)$ for $\alpha \geq \beta$. Now $||g|| \leq \sup ||f_k|| \leq 1$, so $\lim_{\alpha<\beta} g(\alpha)$ exists, possibly not equal to $g(\beta)$. Let $g_1(\alpha) = g(\alpha)$ for $\alpha \neq \beta$, $g_1(\beta) = \lim_{\alpha<\beta} g(\alpha)$. Then $g_1 \in B$. Note that $g_1 = f_k$ on $[0, \alpha_{k-1}]$, $g_1(\omega_1) = f_k(\omega_1)$.

Now consider $h_k = g_1 + f_k - f_{k+1}$. Then $h_k \in 3B$. Also $g_1 = f_{k+1}$ on $[0, \alpha_k]$, so $h_k = f_k$ on $[0, \alpha_k]$. Thus $h_k \in U_k$. Also, $\lim_k h_k(\alpha) = g_1(\alpha)$ for all α. This shows that every neighborhood if g_1 in $3B$ meets infinitely many U_k's, so $\mathcal{U}$ is not discrete on $3B$. □

5.2 Corollary. There is an uncountable discrete family of open sets in B. Therefore, there is no (weakly) continuous retraction of $3B$ onto B, and in particular, there is no retraction of X onto B.

Proof. If $0 < \alpha < \omega_1$, let

$$V_\alpha = \{f \in B : f(\alpha) < \frac{1}{10}, f(\alpha+1) > \frac{9}{10}\} .$$

Then $\mathcal{U} = \{V_\alpha : 0 < \alpha < \omega_1\}$ is an uncountable discrete family of open sets in B. □

The problem of finding retractions onto the unit ball has been studied by Wheeler [10].

If $X = J(\omega_1)$ is the long James space, it is proved in [6] that X is not realcompact. This is done as follows. Identifying X^{**} with $\tilde{J}(\omega_1)$, we may define $F \in X^{**}$ by :

$$\text{(1)} \qquad F(\alpha) = 0 \text{ for } \alpha < \omega_1 , \ F(\omega_1) = 1 .$$

It is easily seen from Corson's criterion that $F \in \upsilon X$, but F is not continuous at ω_1, so $F \notin X$. Thus X is not realcompact. Note that $||F|| = 1$, so $F \in B_{X^{**}}$. But F cannot be used to show that B is not realcompact, as the following result shows. The wording is somewhat awkward because it is not clear that υB can be identified with a subset of X^{**}; certainly the inclusion $B \to X$ extends to a canonical map $\upsilon B \to \upsilon X \subsetneq X^{**}$.

5.3 THEOREM. Let $X = J(\omega_1)$. There is no element of υB whose image in υX is F defined in (1).

Proof. Let (f_ξ) be a σ-net in B, suppose $f_\xi(\alpha) \to 0$ for $\alpha < \omega_1$ and $f_\xi(\omega_1) \to 1$. I will show that there is $h \in C(B)$ such that $h(f_\xi)$ does not converge. This suffices to prove the result, as noted in Section 2.

By taking a cofinal subset of the directed set, we may assume $f_\xi(\omega_1) = 1$ for all ξ. Also, $f_\xi(0) = 0$ for all ξ and $||f_\xi|| \leq 1$, so $0 \leq f_\xi(\alpha) \leq 1$ for all ξ and all $\alpha \in [0, \omega_1]$. Let

$$P_{\alpha,\varepsilon} = \{f \in B : f = 0 \text{ on } [0, \alpha], f(\alpha + 1) > \varepsilon\}.$$

Then

$$f_\xi \in \bigcup_{n=1}^{\infty} \bigcup_{\alpha<\omega_1} P_{\alpha,1/n}$$

for all ξ, so (again taking a cofinal subset) we may assume

$$f_\xi \in \bigcup_{\alpha<\omega_1} P_{\alpha,\varepsilon}$$

for some fixed $\varepsilon > 0$. That is, for every ξ there exists $\alpha_\xi < \omega_1$, such that $f_\xi = 0$ on $[0, \alpha_\xi]$ and $f(\alpha_\xi + 1) > \varepsilon$. Given this ε, choose $\delta > 0$ so small that $3\delta < \varepsilon$ and $(\varepsilon - 2\delta)^2 + (1 - 2\delta)^2 > 1$.

For each $\alpha < \omega_1$, define

$$U_\alpha = \{f \in B : f(\alpha) < \delta, f(\alpha + 1) > \varepsilon - \delta, f(\omega_1) > 1 - \delta\},$$

$$\overline{U}_\alpha = \{f \in B : f(\alpha) \leq \delta, f(\alpha + 1) \geq \varepsilon - \delta, f(\omega_1) \geq 1 - \delta\},$$

so that $f_\xi \in U_{\alpha_\xi}$. The sets U_α are cozero sets in B. I claim that the $\overline{U}_\alpha$ are disjoint, since δ is so small: indeed, suppose $f \in \overline{U}_\alpha$ are $\beta \geq \alpha + 2$. Then $||f||^2 \geq |f(\beta) - f(\alpha + 1)|^2 + |f(\omega_1) - f(\beta)|^2$, so that if $f(\beta) \leq \delta$, then $||f||^2 \geq (\varepsilon - 2\delta)^2 + (1 - 2\delta)^2 > 1$. Thus $f(\beta) > \delta$, so $f \notin \overline{U}_\beta$. Also, $f \notin \overline{U}_{\alpha+1}$ since $3\delta < \varepsilon$.

Next, I claim that any subcollection $\{\overline{U}_\alpha\}_{\alpha\ A}$ of the $\overline{U}_\alpha$ has closed union. Let g be in the closure of $\bigcup_{\alpha\in A} \overline{U}_\alpha$. Let $\alpha_0 < \omega_1$ be such that g is constant on $[\alpha_0, \omega_1]$. Then g is not close to any member of $\bigcup_{\alpha>\alpha_0, \alpha\in A} \overline{U}_\alpha$, so g is in the closure of $\bigcup_{\alpha\leq\alpha_0, \alpha\in A} \overline{U}_\alpha$. Let β_0 be the smallest ordinal such that g is in the closure of $\overline{\bigcup_{\alpha\leq\beta_0,\alpha\in A}\overline{U}_\alpha}$. If β_0 is a successor ordinal, then $g \in \overline{U}_{\beta_0}$. If β_0 is a limit ordinal, and $g \notin \overline{U}_{\beta_0}$, then continuity of g yields $\beta_1 < \beta_0$ such that $|g(\alpha) - g(\alpha + 1)| < \delta/2$ for $\beta_1 \leq \alpha < \beta$. Then g is not close to any member of $\bigcup_{\beta_1\leq\alpha<\beta,\alpha\in A}\overline{U}_\alpha$, so g is in the closure of $\bigcup_{\alpha\leq\beta_1,\alpha\in A}\overline{U}_\alpha$, a contradiction.

Finally, note that (f_ξ) is a σ-net in the disjoint union $\bigcup_{\alpha<\omega_1} U_\alpha$ and $\aleph_1$ is not a 2-valued measurable cardinal, so there exist disjoint A_1, $A_2 \subseteq [0, \omega_1)$ such that $f_\xi \in \bigcup_{\alpha\in A_1} U_\alpha$ frequently and $f_\xi \in \bigcup_{\alpha\in A_2} U_\alpha$ frequently. By the closedness of the unions above, there is a continuous function $h \in C(B)$ such that $h = 0$ on $\overline{U}_\alpha$ for $\alpha \in A_2$, but

$$h(f) = (\delta - f(\alpha))(f(\alpha + 1) - \varepsilon + \delta)(f(\omega_1) - 1 + \delta)$$

on $\overline{U}_\alpha$ for $\alpha \in A_1$. Thus $h(f_\xi) = 0$ frequently and $h(f_\xi) > \delta^3$ frequently. So $h(f_\xi)$ does not converge. □

It should be remarked that the above result shows that Corson's criterion for elements of υX fails to characterize υB.

Even though F cannot be used to prove it, the ball B of $X = J(\omega_1)$ is not realcompact. We can use a small multiple of F for this.

<u>5.4 THEOREM.</u> There is an element of υB whose image in υX is $(.1)F$.

<u>Proof.</u> For countable ordinal α, let $f_\alpha = (.1)\chi_{(\alpha,\omega_1]}$. I will show that $h(f_\alpha)$ converges for all $h \in C(B)$. This will mean that $(.1)F = \lim_\alpha f_\alpha$ is in (the image of) υB but not in B, so B is not realcompact.

Let $h \in C(B)$. Suppose (for purposes of contradiction) that $h(f_\alpha)$ does not converge. Then (by uncountable confinality) there exist $a < b$ such that $h(f_\alpha) > b$ frequently and $h(f_\alpha) < a$ frequently. Let $A_1 = \{\alpha : h(f_\alpha) > b\}$, $A_2 = \{\alpha : h(f_\alpha) < a\}$. Both are uncountable. For each $\alpha \in A_1$, choose an open neighborhood U_α of f_α determined by finitely many functionals: these functionals involve only countably many coordinates, say $K_\alpha = [0, \overline{\alpha}] \cup \{\omega_1\}$, where $\overline{\alpha} < \omega_1$. Thus if $f|_{K_\alpha} = f_\alpha|_{K_\alpha}$, then $h(f) > b$.

Similarly, for $\alpha \in A_2$ we get $K\alpha = [0, \bar{\alpha}] \cup \{\omega_1\}$, where $\bar{\alpha} < \omega_1$, and if $f|_{K_\alpha} = f_\alpha|_{K_\alpha}$ then $h(f) < a$.

Now define inductively $\alpha_1 < \alpha_2 < \ldots$ so that $\alpha_{k+1} > \bar{\alpha}_k$, $\alpha_k \in A_1$, for odd k, $\alpha_k \in A_2$ for even k. Pick $\beta > \sup_k \bar{\alpha}_k$, $\beta < \omega_1$. Then let

$$g_k = (.1)\chi_{(\alpha_k, \bar{\alpha}_k]} + (.1)\chi_{(\beta, \omega_1]} .$$

then $g_k|_{K\alpha_k} = f_{\alpha_k}|_{K_{\alpha_k}}$ so $h(g_k) > b$ for k odd, $h(g_k) < a$ for k even. But g_k converges weakly to $(.1)\chi_{(\beta,\omega_1]}$, and this contradicts the continuity of h. □

6. The next example is $X = \ell_\infty$. This space is realcompact, but not measure-compact. One way to see that X is not measure-compact is based on an observation of Hagler (see [2, p. 43]). He exhibits a function $\phi : [0, 1] \to \ell_\infty$ which is scalarly measurable (and thus Baire measurable [4, Theorem 2.3]), but not scalarly equivalent to a Bochner measurable function, so that the image of Lebesgue measure under ϕ is not a τ-smooth measure (see [4, Section 5]).

Now Hagler's function has range in B_X, so in order to show that B_X is not measure-compact, it is enough to show that ϕ is Baire measurable into B_X. That is, if $h \in C(B_X)$, then $h \circ \phi$ is Lebesgue measurable. This can be done. But my proof is so long, and the result apparently so useless, that I will not include it here. Let me include only the following hints.

Suppose $h \circ \phi$ is not Lebesgue measurable. Then (restricting to a subset of positive measure) there exist $a < b$ so that on every set of positive measure, $h \circ \phi$ has values $> b$ and values $< a$. Something like the constructions in Theorems 4 and 5.4 can then be carried out (on branches of a binary tree) to find points $t_k \in [0, 1]$ and $t^* \in [0, 1]$ so that $y_k = \phi(t^*) + \phi(t_k) - \phi(t_{k+1})$ converges weakly to $\phi(t^*)$, but $h(y_k) > b$ for odd k, $h(y_k) < a$ for even k. This contradicts the continuity of h.

References

1. H.H. Corson, The weak topology of a Banach space, Trans. Amer. Math Soc. 101 (1961) 1-15.

2. J. Diestel and J.J. Uhl, Vector Measures, American Mathematical Society, 1977.

3. N. Dunford and J.T. Schwartz, Linear Operators I, Interscience Publishers, 1957.

4. G.A. Edgar, Measurability in a Banach space, Indiana Univ. Math. J. 26(1977) 663-677.

5. G.A. Edgar, Measurability in a Banach space II, Indiana Univ. Math. J. 28(1979) 559-579.

6. G.A. Edgar, A long James space, In: Measure Theory, Oberwolfach 1979, D. Kölzow (editor), Lecture Notes in Mathematics 794, Springer-Verlag, 1980, pp. 31-36.

7. G.A. Edgar and R.F. Wheeler, Topological properties of Banach spaces. Preprint.

8. L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, 1960.

9. W. Moran, Measures and mappings on topological spaces, Proc. London Math. Soc. 19(1969) 493-508.

10. R. Wheeler, The retraction property, CCC property, and Dunford-Pettis-Phillips property for Banach spaces, Measure Theory, Oberwolfach 1981, D. Kölzow (editor), Lecture Notes in Mathematics, 945, Springer-Verlag, 1982, pp. 252-262.